Casimir interaction: pistons and cavity

# Casimir interaction: pistons and cavity

**Valery N Marachevsky**

Laboratoire Kastler Brossel, CNRS, ENS, UPMC, Campus Jussieu case 74, 75252 Paris, France
and
V A Fock Institute of Physics, St. Petersburg State University, 198504 St. Petersburg, Russia

E-mail: maraval@mail.ru

**Abstract**
The energy of a perfectly conducting rectangular cavity is studied by making
use of piston interactions. The exact solution for a 3D perfectly conducting
piston with an arbitrary cross section is discussed.

PACS numbers: 12.20.−m, 03.70.+k

## 1. Introduction

The Casimir effect [1] was studied in various specific cases and geometries [2, 3]. A new
geometry that recently attracted attention in the theory of the Casimir effect is the piston
geometry.

A piston plate is perpendicular to the walls of a semi-infinite cylinder and moves freely
inside it, this geometry was first investigated in a 2D Dirichlet model [4].

An exact solution for a perfectly conducting square piston at zero temperature was found
in a 3D model in the electromagnetic and scalar cases [5] by making use of a geometric
optics approach; the limit of short distances between the piston and the base of a cylinder
was found in [5, 6] for an arbitrary cross section of a piston; rectangular geometries and finite
temperatures were considered in [6].

In this paper and our previous related papers [7–9] we considered a slightly different
geometry—two piston plates inside an infinite cylinder, which yielded the same results for
rectangular pistons as in the case of a semi-infinite cylinder due to perfectly conducting
boundary conditions. In [7–9] an exact solution for arbitrary cross sections and arbitrary
distances between piston plates was found at zero and finite temperatures in the electromagnetic
3D case. Rectangular and circular cross sections are special cases of our general solution.

A dilute circular piston and cylinder were studied perturbatively in [10]. In this case the
force on two plates inside a cylinder and the force in a piston geometry differ essentially. The
force in a piston geometry can change sign in this approximation for thin enough walls of
the material.

Different examples of pistons in a scalar case were investigated in [11–13].

The case when the piston's cross section differs from that of a cylinder was recently studied numerically [14] and by means of a geometric optics approach [15].

Throughout this paper, we study the Casimir energies of an electromagnetic field with perfectly conducting boundary conditions imposed. First we study the energy of a rectangular cavity by making use of piston interactions. Then we generalize the formulae for the case of a 3D piston with an arbitrary cross section and consider several special and limiting cases. Our formulae can be applied in every case when the two-dimensional Dirichlet and Neumann boundary problems for Helmholtz equation can be solved analytically or numerically. The formulae of section 2 (9)–(12), (14) and (15) are new. The formulae of section 2 (8), (13) were derived in author's lectures [7] and have not been published before, the formulae of section 3 were derived by the author [7–9], mathematical details can be found in [8, 9]. The central new result of this paper is the formula (15).

We take $\hbar = c = 1$.

## 2. Construction of a rectangular cavity

The Casimir energy can be regularized as follows:

$$E = \frac{1}{2} \sum_{\omega_l} \omega_l^{-s}, \tag{1}$$

where $s$ is large enough to make (1) convergent. Then it should be continued analytically (1) to the value $s = -1$, this procedure yields the renormalized finite Casimir energy. The regularized electromagnetic Casimir energy for the rectangular cavity $E_{\text{cavity}}(a, b, c, s)$ can be written in terms of Epstein $Z_3\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}; s\right)$ and Riemann $\zeta_R(s)$ zeta functions [2]

$$E_{\text{cavity}}(a, b, c, s) = \frac{\pi}{8} \left( \sum_{n_1,n_2,n_3=-\infty}^{+\infty}{}' \left[ \left(\frac{n_1}{a}\right)^2 + \left(\frac{n_2}{b}\right)^2 + \left(\frac{n_3}{c}\right)^2 \right]^{-s/2} \right.$$

$$\left. - 2\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \sum_{n=1}^{+\infty} \frac{1}{n^s} \right) \tag{2}$$

$$Z_3\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}; s\right) = \sum_{n_1,n_2,n_3=-\infty}^{+\infty}{}' \left[ \left(\frac{n_1}{a}\right)^2 + \left(\frac{n_2}{b}\right)^2 + \left(\frac{n_3}{c}\right)^2 \right]^{-s/2} \tag{3}$$

$$\zeta_R(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}. \tag{4}$$

The prime means that the term with all $n_i = 0$ should be excluded from the sum. The reflection formulae for an analytical continuation of zeta functions exist

$$\Gamma\left(\frac{s}{2}\right) \pi^{-s/2} \zeta_R(s) = \Gamma\left(\frac{1-s}{2}\right) \pi^{(s-1)/2} \zeta_R(1-s) \tag{5}$$

$$\Gamma\left(\frac{s}{2}\right) \pi^{-s/2} Z_3(a, b, c; s) = (abc)^{-1} \Gamma\left(\frac{3-s}{2}\right) \pi^{(s-3)/2} Z_3\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}; 3-s\right). \tag{6}$$

The renormalized Casimir energy for a perfectly conducting rectangular cavity can therefore be written as [16]

$$E_{\text{cavity}}(a, b, c) = -\frac{abc}{16\pi^2} Z_3(a, b, c; 4) + \frac{\pi}{48}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right). \tag{7}$$
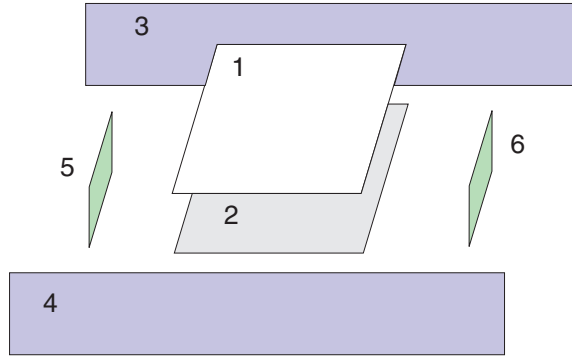
**Figure 1.** Construction of a cavity.

(This figure is in colour only in the electronic version)

The expression (7) can be rewritten in a different mathematical form [7]

$$
E_{\text{cavity}}(a, b, c) = \frac{\pi}{48a} + \frac{\pi}{48b} + \frac{\pi}{48c} + a E_{\text{waveguide}}(b, c)
$$

$$
+ \frac{1}{4} \sum_{n_2, n_3 = -\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\mathrm{d}p}{2\pi} \ln\left( 1 - \exp\left[ -2a \sqrt{\left(\frac{\pi n_2}{b}\right)^2 + \left(\frac{\pi n_3}{c}\right)^2 + p^2} \right] \right). \quad (8)
$$

Here the Casimir energy for a unit length of a rectangular waveguide is ($t = b/c$ or $t = c/b$)

$$
E_{\text{waveguide}}(b, c) = E_{\text{waveguide}}(c, b)
$$

$$
= -\frac{\pi^2}{720t^2 bc} + \frac{t}{4\pi bc} \sum_{n=-\infty}^{+\infty} \int_0^{+\infty} \mathrm{d}p \, p \ln\left( 1 - \exp\left( -2\sqrt{\frac{\pi^2 n^2}{t^2} + p^2} \right) \right)
$$

$$
= -\frac{\pi^2}{720t^2 bc} - \frac{1}{bc} \sum_{n=1}^{+\infty} \left( \frac{\csc^2(n\pi/t)}{16n^2} + \frac{t \coth(n\pi/t)}{16\pi n^3} \right). \quad (9)
$$

Let us discuss different terms appearing in (8) and thus clarify the physical meaning of a zeta function regularization in this case.

Imagine that a piston is large in two dimensions with sides $a_0$ and $c_0$ (plates 1 and 2 in figure 1). In this case the contribution to the energy in this geometry is given by the Casimir result for two parallel plates with the surface area $a_0 c_0$ and the edge term $\pi/48b$

$$
-\frac{\pi^2 a_0 c_0}{720 b^3} + \frac{\pi}{48b}, \quad (10)
$$

where one of the terms in $a_0 E_{\text{waveguide}}(b, c_0)$ ($t = b/c_0$) is taken into account.

The next step is to move other pistons (plates 3 and 4 in figure 1) that have a large side $a_0$ between these already existing parallel plates. The energy change is equal to

$$
a_0 E_{3-4}(b, c) + \frac{\pi}{48c}, \quad (11)
$$

where

$$
E_{3-4}(b, c) = -\frac{1}{bc} \sum_{n=1}^{+\infty} \left( \frac{\csc^2(n\pi/t)}{16n^2} + \frac{t \coth(n\pi/t)}{16\pi n^3} \right). \quad (12)
$$

In (11) another term in $a_0 E_{\text{waveguide}}(b, c)$ was taken into account ($c$ is a distance between the two piston plates 3 and 4, $t = b/c = c/b$). From the energy change (11) it is straightforward

to obtain the force on the piston plates 3 and 4, and in the limit $a_0 \to \infty$ one immediately obtains the exact force on a unit length of stripes 3 and 4 from $E_{3-4}(b, c)$.

The expression for the energy change (10) is valid only when $b$ is much smaller than the sizes of the plates 1, 2, and (11) is valid when $c$ is much smaller than $a_0$ and of the order $b$ or less.

Some comments are needed to clarify the meaning of the terms $\pi/48b + \pi/48c$ in the energy expression (8). These terms appear due to edges of the piston. They are precisely equal to next to leading order terms in the expansion (26), which means that they account for four rectangular edges ($\chi = 1/4$) of the finite piston in two different expansions (for small $b$ and small $c$).

The next possible step is to insert pistons 5 and 6 from the opposite sides of the existing waveguide with sides $b$ and $c$ and move them towards each other. The term [7]

$$E_{5-6}(a, b, c) = \frac{1}{4} \sum_{n_2,n_3=-\infty}^{+\infty}{}' \int_{-\infty}^{+\infty} \frac{\mathrm{d}p}{2\pi} \ln\left(1 - \exp\left[-2a\sqrt{\left(\frac{\pi n_2}{b}\right)^2 + \left(\frac{\pi n_3}{c}\right)^2 + p^2}\right]\right) \quad (13)$$

yields the interaction energy of two plates distance $a$ apart inside an infinite rectangular cylinder with sides $b$ and $c$ (the term $n2 = n3 = 0$ is excluded from the sum due to a cancellation with the term $\pi/(48a)$ in (8)). The force on the pistons 5 and 6 is straightforward.

In summary, the expression

$$\Delta E = E_{\text{cavity}}(a, b, c) - \frac{\pi}{48b} - \frac{\pi}{48c} + \frac{ac}{a_0 c_0}\frac{\pi}{48b} + \frac{a}{a_0}\frac{\pi}{48c} \quad (14)$$

yields the energy change inside a cavity volume with sides $(a, b, c)$ during a construction of the following system:

*Step* 1. pistons 1 and 2 are being moved inside a waveguide with large sides $a_0, c_0$ from a large distance between them towards each other until the distance $b$ between them is achieved. For a validity of the energy change (10) it is necessary to assume $b \ll a_0, b \ll c_0$ (note that in (14) we consider the energy change inside the volume with sides $(a, b, c)$, which is inside the volume $(a_0, b, c_0)$; its energy is $ac/a_0 c_0$ times less than the energy of the volume $(a_0, b, c_0)$—this is why the expression (10) is multiplied by a factor $ac/a_0 c_0$ in (14)).

*Step* 2. pistons 3 and 4 with sides $a_0, b$ are being moved inside the existing box $(a_0, b, c_0)$ between the existing pistons 1 and 2 towards each other until the distance $c$ between them is achieved. For a validity of the energy change (11) it is necessary to assume $c \ll a_0$, also $c \ll c_0$.

*Step* 3. pistons 5 and 6 with sides $b, c$ are being moved inside the existing box $(a_0, b, c)$ towards each other until the distance $a$ between them is achieved. It is assumed here that $a \ll a_0$. However, the formula (13) itself is exact for arbitrary values of $a$ in the limit $a_0 \to \infty$, i.e. for plates inside an infinite waveguide.

During each step the energy in the system decreases, so the force between moving pistons is always attractive.

In the limit of infinite plates 1, 2 and stripes 3, 4 ($a_0, c_0 \to \infty$) the total energy change (steps 1–3) inside the cavity volume with sides $(a, b, c)$ can be written in the form

$$\Delta E_{\text{tot}} = -\frac{abc}{16\pi^2} Z_3(a, b, c; 4) + \frac{\pi}{48a}. \quad (15)$$

The Casimir energy of the cavity (15) depends on the process of its construction, it is not symmetrical in $a, b, c$ as was generally thought before (compare (7) and (15)). This is the central result of our paper.
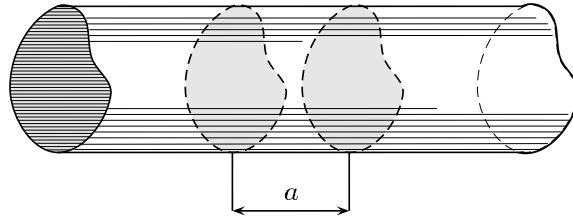
4

**Figure 2.** Two plates inside an infinite cylinder.

## 3. Arbitrary cross section results

Suppose there are two plates inside an infinite cylinder of an arbitrary cross section $M$ (figure 2). To calculate the force on each plate imagine that four parallel plates are inserted inside an infinite cylinder and then two exterior plates are moved to spatial infinity. This situation is exactly equivalent to three perfectly conducting cavities touching each other. From the energy of this system one has to subtract the Casimir energy of an infinite cylinder without plates inside it. Doing so we obtain the energy of interaction between the interior parallel plates and the attractive force on each interior plate inside the cylinder [7–9]

$$\mathcal{E}(a) = \sum_{\omega_{\text{wave}}} \frac{1}{2} \ln(1 - \exp(-2a\omega_{\text{wave}})) \tag{16}$$

$$F(a) = -\frac{\partial \mathcal{E}(a)}{\partial a}, \tag{17}$$

the sum here is over all $TE$ and $TM$ eigenfrequencies $\omega_{\text{wave}}$ for a cylinder with the cross section $M$ and an infinite length.

In fact, the Casimir energy of our electromagnetic system is proportional to the sum of free energies for two boson scalar fields (with Dirichlet and Neumann boundary conditions imposed at the boundary of an infinite cylinder with a cross section $M$ and zero Neumann eigenvalue excluded) at finite temperature $T = 1/\beta$ if we make a substitution $a \to \beta/2$. Free energies have a well-defined finite part, their sum up to a factor $1/a$ coincides with (16).

One can rewrite (16) in a different form [9, 17]

$$\mathcal{E}(a) = -\frac{1}{2\pi} \sum_{l=1}^{+\infty} \left( \sum_{\lambda_{kD}} \frac{\lambda_{kD} K_1(2l\lambda_{kD}a)}{l} + \sum_{\lambda_{iN}} \frac{\lambda_{iN} K_1(2l\lambda_{iN}a)}{l} \right). \tag{18}$$

Here $K_1$ is a modified Bessel function and

$$\Delta^{(2)} f_k(x, y) = -\lambda_{kD}^2 f_k(x, y) \qquad f_k(x, y)|_{\partial M} = 0 \tag{19}$$

$$\Delta^{(2)} g_i(x, y) = -\lambda_{iN}^2 g_i(x, y) \qquad \left. \frac{\partial g_i(x, y)}{\partial n} \right|_{\partial M} = 0. \tag{20}$$

Our results are *exact for an arbitrary curved geometry of a cylinder*.

For a rectangular cylinder with sides $b$ and $c$ the exact Casimir energy of two plates inside it can be written as [9]

$$\mathcal{E}_{\text{rect}}(a) = -\sum_{l=1}^{+\infty} \sum_{m,n=-\infty}^{+\infty}{}' \frac{\sqrt{m^2/b^2 + n^2/c^2}}{4l} K_1(2l\pi a\sqrt{m^2/b^2 + n^2/c^2}). \tag{21}$$

The term $m = n = 0$ is omitted in the sum.

For a circular cylinder the eigenvalues of the two-dimensional Helmholtz equation $\lambda_{kD}, \lambda_{iN}$ are determined by the roots of Bessel functions and derivatives of Bessel functions. The exact Casimir energy of two circular plates of radius $b$ separated by a distance $a$ inside an infinite circular cylinder of radius $b$ is given by [9]

$$\mathcal{E}_{\mathrm{circ}}(a) = -\sum_{l=1}^{+\infty}\sum_{\nu=0}^{+\infty}\sum_j \frac{1}{2\pi b}\frac{\mu_{D\nu j}K_1(2l\mu_{D\nu j}a/b) + \mu_{N\nu j}K_1(2l\mu_{N\nu j}a/b)}{l}, \tag{22}$$

$$J_\nu(\mu_{D\nu j}) = 0, \qquad J'_\nu(\mu_{N\nu j}) = 0.$$

The sum is over positive $\mu_{D\nu j}$ and $\mu_{N\nu j}$.

The leading asymptotic behaviour of $\mathcal{E}(a)$ for long distances $\lambda_{1D}a \gg 1, \lambda_{1N}a \gg 1$ is determined by the lowest positive eigenvalues of the two-dimensional Dirichlet and Neumann problems $\lambda_{1D}, \lambda_{1N}$

$$\mathcal{E}(a)|_{\lambda_{1D}a\gg 1,\,\lambda_{1N}a\gg 1} \sim -\frac{1}{4\sqrt{\pi a}}\left(\sqrt{\lambda_{1D}}\,\mathrm{e}^{-2\lambda_{1D}a} + \sqrt{\lambda_{1N}}\,\mathrm{e}^{-2\lambda_{1N}a}\right), \tag{23}$$

so the Casimir force between the two plates in a cylinder is exponentially small for long distances. This important property of the solution is due to the gap in the frequency spectrum or, in other words, it is due to the finite size of the cross section of the cylinder. Due to this property one needs a finite number of the eigenvalues of the Helmholtz equation for $2D$ Dirichlet and Neumann boundary problems (19)–(20) to obtain the Casimir energy at a specific distance $a$ between the plates with a desired accuracy.

The free energy at a temperature $T = 1/\beta$ describing the interaction of two parallel perfectly conducting plates inside an infinite perfectly conducting cylinder with the cross section $M$ has the form [7–9]

$$\mathcal{F}(a, \beta) = \frac{1}{\beta}\sum_{\lambda_{kD}}\sum_{m=-\infty}^{+\infty}\frac{1}{2}\ln\left(1 - \exp\left(-2a\sqrt{\lambda_{kD}^2 + p_m^2}\right)\right)$$

$$+ \frac{1}{\beta}\sum_{\lambda_{iN}}\sum_{m=-\infty}^{+\infty}\frac{1}{2}\ln\left(1 - \exp\left(-2a\sqrt{\lambda_{iN}^2 + p_m^2}\right)\right). \tag{24}$$

Note that $\lambda_{iN} \neq 0$. We used the standard notation $p_m = 2\pi mT$.

In the long distance limit $a \gg \beta/(4\pi)$ one has to keep only $m = 0$ term in (24). Thus the free energy of the plates inside a cylinder in the high temperature limit is equal to [8]

$$\mathcal{F}(a, \beta)|_{a\gg\beta/(4\pi)} = \frac{1}{2\beta}\sum_{\lambda_{kD}}\ln\left(1 - \exp(-2a\lambda_{kD})\right) + \frac{1}{2\beta}\sum_{\lambda_{iN}}\ln\left(1 - \exp(-2a\lambda_{iN})\right). \tag{25}$$

One can check that the limit $\lambda_{1D}a \ll 1, \lambda_{1N}a \ll 1$ in (25) immediately yields the known high temperature result for two parallel perfectly conducting plates separated by a distance $a$ [18].

For $a \ll \beta/(4\pi)$ and $\lambda_{1D}a \ll 1, \lambda_{1N}a \ll 1$ one can use the heat kernel expansion [19, 20] and properties of the zeta function [21–23] to obtain the leading terms for the free energy [9]

$$\mathcal{F}(a, \beta)|_{a\ll\beta/(4\pi),\,\lambda_{1D}a\ll 1,\,\lambda_{1N}a\ll 1} = -\frac{\zeta_R(4)}{8\pi^2}\frac{S}{a^3} + \frac{\zeta_R(2)}{4\pi a}(1 - 2\chi) + O(1), \tag{26}$$

where

$$\chi = \sum_i \frac{1}{24}\left(\frac{\pi}{\alpha_i} - \frac{\alpha_i}{\pi}\right) + \sum_j \frac{1}{12\pi}\int_{\gamma_j} L_{aa}(\gamma_j)\,\mathrm{d}\gamma_j. \tag{27}$$

Here $S$ is an area of the cross section $M$, $\alpha_i$ is the interior angle of each sharp corner at the boundary $\partial M$ and $L_{aa}(\gamma_j)$ is the curvature of each boundary smooth section described by the curve $\gamma_j$. The force calculated from (26) coincides with the zero temperature force $F_C$ in [5], (equation 7).

## Acknowledgments

## References

[1]  Casimir H B G 1948 *Proc. K. Ned. Akad. Wet.* **51** 793
[2]  Bordag M, Mohideen U and Mostepanenko V M 2001 *Phys. Rep.* **353** 1
[3]  Nesterenko V V, Lambiase G and Scarpetta G 2004 *Riv. Nuovo Cimento* **27** (6) 1
     Milton K A 2004 *J. Phys. A: Math. Gen.* **37** R209
     Lambrecht A, Maia Neto P A and Reynaud S 2006 *New J. Phys.* **8** 243
[4]  Cavalcanti R M 2004 *Phys. Rev.* D **72** 065015
[5]  Hertzberg M P, Jaffe R L, Kardar M and Scardicchio A 2005 *Phys. Rev. Lett.* **95** 250402
[6]  Hertzberg M P, Jaffe R L, Kardar M and Scardicchio A 2007 Casimir forces in a piston geometry at zero and finite temperature *Preprint* 0705.0139 [quant-ph]
[7]  Marachevsky V N 2005 One loop boundary effects: techniques and applications *Preprint* hep-th/0512221
[8]  Marachevsky V N 2006 Casimir energy of two plates inside a cylinder *Proc. 14th Int. Sem. High Energy Phys. QUARKS-2006 (St. Petersburg, Russia, 19–25 May 2006) Preprint* hep-th/0609116
[9]  Marachevsky V N 2007 *Phys. Rev.* D **75** 085019
[10] Barton G 2006 *Phys. Rev.* D **73** 065018
[11] Fulling S A and Wilson J H 2007 *Phys. Rev.* A **76** 012118
[12] Edery A 2006 *J. Phys. A: Math. Gen.* **39** 685
     Edery A 2007 *Phys. Rev.* D **75** 105012
     Edery A and MacDonald I 2007 *J. High Energy Phys.* JHEP09(2007)005
[13] Zhai X-h and Li X-z 2006 Casimir pistons with hybrid boundary conditions *Preprint* hep-th/0612155
[14] Rodriguez A, Ibanescu M, Ianuzzi D, Capasso F, Joannopoulos J D and Johnson S G 2007 *Phys. Rev. Lett.* **99** 80401
[15] Zaheer S, Rodriguez A W, Johnson S G and Jaffe R L 2007 Optical-approximation analysis of sidewall-spacing effects on the force between two squares with parallel sidewalls *Preprint* 0709.0699 [quant-ph]
[16] Lukosz W 1971 *Physica (Amsterdam)* **56** 109
[17] Nesterenko V V and Pirozhenko I G 1997 *J. Math. Phys.* **38** 6265
[18] Sauer F 1962 *PhD Thesis* (Gottingen)
     Brevik I, Ellingsen S A and Milton K A 2006 *New J. Phys.* **8** 236
[19] Vassilevich D V 2003 *Phys. Rep.* **388** 279
[20] Gilkey P B 1994 *Invariance Theory, the Heat Equation, and the Atiyah-Singer Index Theorem* (Boca Raton, FL: CRC Press)
[21] Kirsten K 2002 *Spectral Functions in Mathematics and Physics* (Boca Raton, FL: CRC Press)
[22] Santangelo E M 2001 *Teor. Mat. Fiz.* **131** 98 (*Preprint* hep-th/0104025)
     Santangelo E M 2002 *Theor. Math. Phys.* **131** 527 (Engl. Transl.)
[23] Elizalde E, Odintsov S D, Romeo A, Bytsenko A and Zerbini S 1994 *Zeta Regularization with Applications* (Singapore: World Scientific)